Analytic dynamics of one-dimensional particle system with strong interaction

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Abstract

We study here the small time dynamics of N electrons on the circle with Coul;omb repulsive interaction and study the series for the velocities (initially zero). The main result is the estimates of the convergence radius from below. We explain how this result is related to the problem of very fast establishing of direct electric current.

1 Introduction

Model and results We consider a system of N point particles $i = 1, 2, ..., N$ on the interval $[0, L) \in R$, with periodic boundary conditions, that is on the circle S_L of length L. Initially, they are situated at the points

$$0 = x_1(0) < ... < x_N(0) < L$$

The trajectories $x_i(t)$ are defined by the following system of N equations

$$\frac{d^2 x_i}{dt^2} = -\frac{\partial U}{\partial x_i} + F(x_i) \tag{1}$$

where the interaction between the particles is

$$U(\{x_i\}) = \sum_{<i,i-1>} V(x_i - x_{i-1})$$

where summations is over all pairs of nearest neighbors on the circle. The Coulomb potential $V(x) = V(-x) = \frac{1}{r}, r = |x|$ is assumed, and we denote $f(r) = -\frac{dV(r)}{dr} = r^{-2}$ the interaction force. Note that the potential is repulsive, infinite at zero and thus, the particles cannot change their order . $F(x)$ is the external force.

In the papers [11, 12] we considered fixed points of such systems. Dynamics is more complicated to understand. It is standard that the solution of the system (1) exists on all time interval and is unique (for any initial conditions), however to get more detailed information about particle trajectories (for sufficiently large N) is sufficiently difficult and demands elaboration of special methods. If moreover F is analytic, then it is well-known [1], that the solution can be presented as the convergent power series in t in some neighborhood of $t = 0$.

Here we consider natural initial conditions, that is for all i

$$\Delta = \Delta_i(0) = x_{i+1}(0) - x_i(0) = \frac{L}{N}, v_i(0) = 0 \tag{2}$$

and moreover, it is convenient to put $x_1(0) = 0$. Note that this configuration is a fixed point for zero external force. We are looking for the solution of the form

$$v_i(t) = \sum_{j=1}^{\infty} c_{i,j} t^j, c_{i,j} = c_{i,j}^{(N)} \tag{3}$$

and get bounds for the convergence radius, dependent on N, of these series under the initial conditions (2).

Theorem 1 *Let F be analytic on the circle S_L. Then*

1. *for $j = 1, 2, ...,$ there exist numbers $b_j < \infty$, not depending on N and such, that for all i, j and any N*

$$|c_{ij}| < b_j N^{\frac{j-1}{2}}$$

2. *if moreover, for some constant $C_F > 0$ and all x and k*

$$|F^{(k)}(x)| \leq C_F^{k+1},$$

then there exists constant $0 < \chi < \infty$, not depending on N and such that for all i, j

$$|c_{ij}| < \chi^j N^{\frac{5}{6}j - \frac{3}{2}}$$

It follows that the convergence radius $R = R(N)$ of the series (3) has a bound from below $R > \chi^{-1} N^{-\frac{5}{6}}$. From the proof of the first assertion of the theorem (see section 2.2) one can see that the bound from above could be of the order $\frac{1}{\sqrt{N}}$, but this not yet rigorously proved. During the proof of the second assertion of the theorem we give the explicit bound for χ. Also, explicit formulas for $c_{ij}, j = 1, 2, 3, 4$ are presented.

Why this model Mathematical problems of statistical physics are elaborated sufficiently deeply for equilibrium systems on the lattice. But on the continuous (euclidean) space only for particle systems with small inverse temperature or small density. There are many other cases where the problems are not even formulated on the mathematical level. One of such cases is the direct electric current. It is described by Ohm's law on the macro level. On the micro level, in any textbook on condensed matter physics, it is described as a flow of free (or almost free) electrons, any of which is accelerated by the external force and impeded by the external media (crystal lattice). Physics and mathematics introduced a lot of such one-particle models with constant accelerating force and various models (about 20, historically the first such model is the famous Drude's model of 1900) of external media, where the particles loose their kinetic energy.

The central question is where the accelerating force comes from. The problem is that the power lines can have hundreds kilometers of length but the external force acts only on the length of several meters of the wire. In fact, there are even more problems with the direct electric current. For example, one should explain why the speed of the flow is permanent and sufficiently slow, but this regime is being established almost immediately. We discussed the first problem in ([10]), but there was no Coulomb interaction there. We shall come back to this problem in the next paper. Here we discuss the second problem.

Consider first a trivial case with constant $F \geq 0$ and with initial conditions (2). Then it is clear that for any i

$$v_i(t) = Ft, x_i(t) = x_i(0) + \frac{Ft^2}{2}$$

that is $x_i(t)$ are analytic in t for any t. Moreover, to reach the speed of order 1 (for example $v_i(t) = 1$) the time of order 1 is necessary. We shall see however that the situation is completely different for non-constant F. Anticipating the events, it is important to note that technically the difference occurs because for constant F all discrete derivatives in the recurrent equations below are identically zero.

If we could prove that the coefficients c_{ij} in the expansion $v_i = \sum_{j=1}^{\infty} c_{i,j} t^j$ grow as N^{aj} for some $a > 0$, then the speed of order 1 will be achieved much earlier - for time of the order $t = N^{-a}$, that is almost immediately. We cannot prove such result as it is but the estimates of the coefficients, obtained here, make such result quite plausible.

The techniques of the paper is apparently new. There are results for small time dynamics of multi-particle systems (see, [2]-[9]) but they are not applicable in our case because of very strong interaction.

2 Proof

2.1 Equations for the coefficients

Fix initial data $x_i(0), v_i(0)$ as in (2) and consider the trajectories $x_i(t) \in S_L$ on the interval $0 \leq t < t_0$ for some $t_0 = t_0(N) > 0$. Putting

$$\Delta_i(t) = x_{i+1}(t) - x_i(t), \Delta = \Delta_i(0) = \frac{L}{N}$$

we have the equations

$$\frac{dv_i}{dt} = f(x_i(t) - x_{i-1}(t)) - f(x_{i+1}(t) - x_i(t)) + F(x_i(t))$$

or

$$\frac{dv_i}{dt} = f(\Delta + \int_0^t [v_i(t_1) - v_{i-1}(t_1)]dt_1) - f(\Delta + \int_0^t [v_{i+1}(t_1) - v_i(t_1)])dt_1) +$$

$$+F(x_i(0) + \int_0^t v_i(t_1)dt_1)$$

Integral equations Equivalent system of integral equations is

$$v_i(t) = \int_0^t [f(\Delta + \int_0^t [v_i(t_1) - v_{i-1}(t_1)]dt_1) -$$

$$- f(\Delta + \int_0^t [v_{i+1}(t_1) - v_i(t_1)]dt_1) + F(x_i(0) + \int_0^t v_i(t_1)dt_1)]dt \tag{4}$$

and can be rewritten as follows

$$v_i(t) = \int_0^t ((\Delta + R_{i-1}(t))^{-2} - (\Delta + R_i(t))^{-2} + F(x_i(0) + \int_0^t v_i(t_1)dt_1))dt \tag{5}$$

where

$$R_{i-1}(t) = \int_0^t (v_i(t_1) - v_{i-1}(t_1))dt_1$$

For the sequel we need some notation for discrete derivatives. Let on the interval $[0, N] \subset Z$ with periodic boundary conditions a function $g(i)$ be given (that is a periodic function on Z with period N). Let us call

$$(\nabla g)(i) = (\nabla^+ g)(i) = g(i+1) - g(i), (\nabla^- g)(i) = g(i) - g(i-1) \tag{6}$$

its right and left derivative correspondingly. Note that they commute and the Leibnitz formula holds

$$\nabla^+(gf)(i) = f(i+1)(\nabla^+ g)(i) + g(i)(\nabla^+ f)(i) = (Sf)(\nabla^+ g) + g(\nabla^+ f) \tag{7}$$

where S is the shift operator

$$(Sf)(i) = f(i+1)$$

Below discrete derivatives will act on the indices i. If the function $f(i)$ does not depend on i, then its (discrete) differentiation gives zero.

We come back to the main equations and rewrite them as follows

$$v_i(t) = \int_0^t dt[(-\nabla^-((\Delta + R_i(t))^{-2}) + F(x_i(0) + \int_0^t v_i(t_1)dt_1)]$$

The following representation of the integrand will be useful

$$(\Delta + R_{i-1}(t))^{-2} - (\Delta + R_i(t))^{-2} + F(x_i(0) + \int_0^t v_i(t_1)dt_1) =$$

$$= \Delta^{-2}(1 + \frac{R_{i-1}}{\Delta})^{-2} - \Delta^{-2}(1 + \frac{R_i}{\Delta})^{-2} + F(x_i(0) + \int_0^t v_i(t_1)dt_1) =$$

$$= F(x_i(0)) + \sum_{m=1}^{\infty} d_m \Delta^{-2-m}(R_{i-1}^m - R_i^m) + [F(x_i(0) + \int_0^t v_i(t_1)dt_1) - F(x_i(0))]$$

where

$$d_m = (-1)^m(m+1)$$

If F is analytic on S_L, then there exists $\epsilon > 0$ sufficiently small and such that for any $x_0 \in S_L$ the following series

$$F(x) = F(x_0) + \sum_{k=1}^{\infty} \frac{F^{(k)}(x_0)}{k!}(x - x_0)^k$$

converges for all $x \in [x_0 - \epsilon, x_0 + \epsilon)$. Then finally

$$v_i(t) = F(x_i(0))t + \int_0^t \sum_{m=1}^{\infty} d_m \Delta^{-2-m}[-\nabla^- R_i^m)]dt + \sum_{k=1}^{\infty} \int_0^t \frac{F^{(k)}(x_i(0))(\int_0^t v_i(t_1)dt_1)^k}{k!} dt \tag{8}$$

Recurrent equations Using (3) and

$$R_{i-1}(t) = \sum_{j=1}^{\infty}(c_{i,j} - c_{i-1,j})\frac{t^{j+1}}{j+1} \tag{9}$$

$$R_i - R_{i-1} = \sum_{j=1}^{\infty}(c_{i+1,j} - 2c_{i,j} + c_{i-1,j})\frac{t^{j+1}}{j+1} \tag{10}$$

we see, substituting (3) to (8), that the right-hand side of (8) can also be presented as a convergent series with well-defined coefficients.

We shall find $c_{i,j}$ by equating the coefficients of t^j. For $j = 1, 2$ the equations give immediately

$$c_{i1} = F(x_i(0)), c_{i,2} = 0 \tag{11}$$

as other summands in the right side of (8) have larger order in t. For $j \geq 3$ the equations for the coefficients of t^j are

$$c_{ij} = \frac{1}{j}[\sum_{m=1}^{\infty} d_m \Delta^{-2-m}(-\nabla^- R_i^m) + \sum_{k=1}^{\infty} \frac{F^{(k)}(x_i(0))(\int_0^t v_i(t_1)dt_1)^k}{k!}]_{j-1} \tag{12}$$

where, for the power series $\phi(t) = \sum_{k=0}^{\infty} a_k t^k$, we denote $[\phi(t)]_j = a_j$. For $j > 2$ the coefficients $c_{i,j}$ can be found recursively, moreover c_{ij} depend only on $c_{i,k}$ with $k \leq j-2$. In fact, the right-hand part of the equation for $c_{i,j}$ does not contain $c_{i,k}$ with $k \geq j-1$, as due to (9), each of c_{ik} appears together with t^{k+1}.

Then the main equations will be

$$c_{ij} = \frac{1}{j} \sum_{m=1}^{\infty} d_m \Delta^{-2-m}(-\nabla^- [(\sum_{j=1}^{\infty}(c_{i+1,j} - c_{i,j})\frac{t^{j+1}}{j+1})^m]_{j-1}) + \tag{13}$$

$$+ \sum_{k=1}^{\infty} \frac{F^{(k)}(x_i(0))}{k!}[\sum_{j=1}^{\infty} c_{i,j}\frac{t^{j+1}}{j+1})^k]_{j-1}$$

We have

$$[(\sum_{j=1}^{\infty} c_{i,i}\frac{t^{j+1}}{j+1})^k]_{j-1} = \sum_{j_1+\ldots+j_m=j-m-1} \frac{c_{i,j_1}}{j_1+1} \ldots \frac{c_{i,j_k}}{j_k+1} \tag{14}$$

where $\sum_{j_1+\ldots+j_m=j-m-1}$ is the sum over all ordered arrays $j_1, ..., j_k$, such that

$$(j_1 + 1) + ... + (j_k + 1) = k + j_1 + ... + j_k = j - 1 \tag{15}$$

It follows

$$k \leq j_1 + ... + j_k = j - 1 - k \leq j - 2, k \leq [\frac{j-1}{2}] \tag{16}$$

Similarly

$$[\sum_{j=1}^{\infty}(c_{i+1,j} - c_{i,j})\frac{t^{j+1}}{j+1})^m]_{j-1} = \sum_{j_1,\ldots,j_m}^{(j-1,m)} \frac{\nabla^+ c_{i,j_1}}{j_1+1} \ldots \frac{\nabla^+ c_{i,j_m}}{j_m+1}$$

and both (15) and (16) hold with k instead of m.

That is why the equations can be written as

$$c = Gc + c^{(0)} \tag{17}$$

where c is the vector $c = \{c_{ij}\}$, the free term $c^{(0)} = \{c_{ij}^{(0)}\}$ is simple

$$c_{i1}^{(0)} = F(x_i(0)), c_{ij}^{(0)} = 0, j \geq 2 \tag{18}$$

and non-linear operator G is defined by

$$c_{i1} = c_{i1}^{(0)} = F(x_i(0)), c_{i2} = 0$$

$$c_{ij} = (Gc)_{ij} = -\sum_{m=1}^{[\frac{j-1}{2}]} \sum_{j_1+...+j_m=j-m-1} A_{ij}(m; j_1, ..., j_m) + \sum_{k=1}^{[\frac{j-1}{2}]} \sum_{j_1+...+j_k=j-k-1} B_{ij}(k; j_1, ..., j_k) \quad (19)$$

for $j \geq 3$, where

$$A_{ij}(m; j_1, ..., j_m) = \frac{1}{j} d_m \Delta^{-2-m} \nabla^{-} \left(\frac{\nabla^{+} c_{i,j_1}}{j_1+1} ... \frac{\nabla^{+} c_{i,j_m}}{j_m+1}\right) \quad (20)$$

$$B_{ij}(k; j_1, ..., j_k) = \frac{1}{j}\frac{1}{k!} F^{(k)}(x_i(0)) \frac{c_{i,j_1}}{j_1+1} ... \frac{c_{i,j_k}}{j_k+1} \quad (21)$$

Further on, $F_{i,k,q}$ will denote any discrete derivative $(\prod_{p=1}^{q} \nabla^{s(p)}) F^{(k)}(x_i(0))$, where $s(p) = \pm$. For the estimates the choice of $s(p)$ does not matter. Put $F_{i,k} = F_{i,k,0}$.

Explicit expression for c_{i3}, c_{i4} is immediately obtained if in the equations (19) we take into account only the terms with $k = 1$ and $m = 1$, as $k.m \leq [\frac{j-1}{2}] \leq 1$)

$$c_{i3} = -\frac{1}{3} d_1 \Delta^{-3} \nabla^{-} \nabla^{+} \frac{c_{i1}}{2} + \frac{1}{3} F^{(1)}(x_i) \frac{c_{i1}}{2} = \frac{1}{6}(d_1 \Delta^{-3} F_{i,0,2} + F_{i,0,0} F_{i,1,0})$$

$$c_{i4} = -\frac{1}{4} d_1 \Delta^{-3} \nabla^{-} (\nabla^{+} \frac{c_{i1}}{2})^2 + \frac{1}{4} F^{(1)}(x_i) \frac{c_{i1}^2}{4} =$$

$$= \frac{1}{8}(-d_1 \Delta^{-3} F_{i,0,2} F_{i,0;1} + \frac{1}{2} F_{i,1,0} F_{i,0,0}^2)$$

From the formulas

$$F_{i,0,1} = F_{i+1,0,0} - F_{i,0,0} = \int_{x_i}^{x_{i+1}} F^{(1)}(x) dx, |F_{i,0,1}| \leq C_F^2 \Delta \quad (22)$$

$$F_{i,0,2} = (F_{i+2,0,0} - F_{i+1,0,0}) - (F_{i+1,0,0} - F_{i,0,0}) = \int_{x_i}^{x_{i+1}} (\int_{x}^{x+\Delta} F^{(2)}(y) dy) dx, |F_{i,0,2}| \leq C_F^3 \Delta^2$$

it follows that

$$|c_{i3}| \leq \frac{1}{3} C_F^3 (\Delta^{-1} + \frac{1}{2}), |c_{i4}| \leq \frac{1}{4} C_F^5 + \frac{1}{16} C_F^4$$

It is easy to see that for most i the coefficients $c_{i,3}$ are really of the order Δ^{-1}.

2.2 Principal exponent

From the recurrent formulas (20) and (21) it follows that c_{ij} are finite and depend on i, j, N. First of all, we shall study them as functions of N for fixed i, j. Otherwise speaking, we shall prove the first part of the theorem. Define the principal exponent

$$I(\xi) = \limsup_{N \to \infty} \frac{\ln |\xi|}{\ln N}$$

for a variable ξ depending on N. Roughly speaking, it shows that the main order of the asymptotics of ξ is $N^{I(\xi)}$.

We shall consider the algebra $\mathbf{A}$ of polynomials of countable number of (commuting) variables $F_{i,k,q}, i = 1, ..., N; k, q = 0, 1, 2, ...$ with real coefficients, not depending on F. For any monomial M of this algebra denote

$$Q(M) = -\sum q$$

over all q in this monomial. The natural mapping of $\mathbf{A}$ onto its subalgebra $\mathbf{A}_0$, generated by all $F_{i,k} = F_{i,k,0}$, is defined by the subsequent substitutions

$$F_{i,k,q} = F_{i+1,k,q-1} - F_{i,k,q-1}$$

or with the following formula

$$(\nabla^+)^n = (S-1)^n = \sum_{k=0}^{n} C_n^k (-1)^k S^{n-k}$$

Lemma 1 *For any monomial $M \in \mathbf{A}$*

$$I(M) \leq Q(M)$$

It is sufficient to prove that

$$I(F_{i,q}) \leq Q(F_{i,q}) = -q$$

This can be done as in (22), using induction in q. Put

$$g_{i,n} = \nabla^n g_i, \Delta = \frac{1}{N}$$

Then

$$g_{i,n+1} = \nabla^{n+1} g_i = g_{i+1,n} - g_{i,n} = \int_{x_i}^{x_i+\Delta} dy_1 (\int_{y_1}^{y_1+\Delta} dy_2 ....(\int_{y_{n-1}}^{y_{n-1}+\Delta} dy_n g^{(n)}(y_n)))$$

and if $|g^{(n)}(x)| \leq C_g^{n+1}$, we have

$$|g_{i,n}| \leq C_g^{n+1} \Delta^n$$

For any polynomial $P = \sum a_r M_r$ with (different) monomials M_r and coefficients a_r, not depending on F, but possibly depending on N, define

$$Q(P) = \max_r (I(a_r) + Q(M_r))$$

in agreement with the previous definition. Then for any polynomial P

$$I(P) \leq \max_r (I(a_r) + I(M_r)) \leq \max_r (I(a_r) + Q(M_r))$$

Note that for any two polyonomials

$$Q(P_1 P_2) \leq Q(P_1) + Q(P_2)$$

Also

$$Q(\nabla^+ P) \leq Q(P) - 1, Q(\nabla^- \nabla^+ P) \leq Q(P) - 2 \quad (23)$$

Denote $\deg P$ the degree of the polynomial $P = \sum a_r M_r$, that is the maximal degree of its monomials.

Lemma 2 *For $j > 2$ c_{ij} is a polynomial in the algebra $\mathbf{A}_0$ and has degree not greater than $j-1$.*

We already saw this for $j = 3, 4$. Note that

$$\deg(\nabla^{\pm} P) = \deg P$$

Now one can use induction: in the formula (20) the degree is $j-2$, and in (21) the degree will be $j-1$.

Note that the recurrent formulas define c_{ij} for all functions $F(x)$, not necessary analytic. That is why the following assertion makes sense.

Lemma 3 *Let F be infinitely differentiable. Then for all i, j*

$$I(c_{ij}) \leq Q(c_{ij}) \leq \frac{j-1}{2}$$

As

$$Q(c_{ij}) = 0, j = 1, 2, Q(c_{i,3}) = 1, Q(c_{i,4}) = 0$$

then the assertion holds for $j = 1, 2, 3, 4$. We shall prove the lemma by induction in j. Assume that

$$Q(c_{ij}) \leq \frac{j-1}{2}$$

for all $j = 1, 2, ..., J-2$.

Then for given $m, j_1, ..., j_m$, accordingly to (15) and (16), we have

$$Q(A_{iJ}(m; j_1, ..., j_m)) \leq 2 + m - 1 + Q(c_{ij_1}) + ... + Q(c_{ij_m}) - m \leq 1 + \frac{1}{2}(j_1 + ... + j_m) - \frac{m}{2} =$$

$$= 1 + \frac{1}{2}(J - m - 1) - \frac{m}{2}$$

as, according to (23), (-1) and $(-m)$ are appended because of the discrete differentiation of the corresponding monomials. The last expression attains its maximum when $m = 1$. It follows

$$Q(A_{iJ}(m; j_1, ..., j_m)) \leq \frac{J-1}{2}$$

Similarly, for $B_{iJ}(k : j_1, ..., j_k)$ we have the following inequalities

$$Q(B_{iJ}(k; j_1, ..., j_k)) \leq \frac{1}{2}(J - 1 - k) - \frac{k}{2} < \frac{J-1}{2}$$

We get thus $Q(c_{iJ}) \leq \frac{J-1}{2}$, and $I(c_{iJ}) \leq Q(c_{iJ}) \leq \frac{J-1}{2}$.

2.3 Convergence radius

Here we shall prove the second assertion of the theorem 1. In the proof it is convenient to write N instead of $\frac{N}{L}$ and assume that $C_F \geq 1$.

We shall use the following majorization principle for infinite system of recurrent equations and inequalities: for example, if two systems of equations are given

$$c_{ij}^{(q)} = P^{(q)}(c_{i1}^{(q)}, ..., c_{i,j-2}^{(q)}), q = 1, 2$$

where $P^{(q)}$ are the polynomials with coefficients $p_{\alpha}^{(q)}$, where $p_{\alpha}^{(2)} \geq 0, |p_{\alpha}^{(1)}| \leq p_{\alpha}^{(2)}$ for all α, and also $|c_{ij}^{(1)}| \leq c_{ij}^{(2)}$ for $j = 1, 2, 3, 4$, then $|c_{ij}^{(1)}| \leq c_{ij}^{(2)}$ for all j. One of such system (we use it in the proof) corresponds to one-particle problem (that is with $N = 1$) with specially chosen external force, will be now introduced. Other auxiliary system $\beta(c_{ij})$ with positive coefficients will be introduced later.

One-particle problem For $j=1,2,...$ and fixed a put

$$g_j=g_j(\frac{a}{2})=\{\frac{a}{2}\}^j\frac{1.3...(2j-1)}{j!}=\{\frac{a}{2}\}^j\frac{(2j)!}{2^j j!j!}\sim\{\frac{a}{2}\}^j\frac{1}{\sqrt{4\pi j}}$$

Then we have

Lemma 4 *For $j=5,6,...$ the following inequalities hold*

$$g_j\geq\frac{1}{j}\sum_{k=1}^{[\frac{j-1}{2}]}\sum_{j_1+...+j_m=j-m-1}(\frac{a}{2})^{k+1}\frac{(k+1)(k+2)}{2}\frac{g_{j_1}}{j_1+1}...\frac{g_{j_k}}{j_k+1}$$

Proof. Let the particle, situated initially at the point $x(0)=0$, move with the speed ($a>0$ being arbitrary)

$$v(t)=\frac{1}{\sqrt{1-at}}=\sum_{j=0}^{\infty}g_j t^j$$

in the field of the external force $F(x)$, which we should find. Then

$$x(t)=\int_0^t v(s)ds=(-\frac{2}{a})\sqrt{1-at}+\frac{2}{a}\Longrightarrow 1-at=(1-\frac{ax}{2})^2$$

$$F=\frac{dv}{dt}=\frac{a}{2}\frac{1}{(1-at)^{\frac{3}{2}}}=\frac{a}{2}\frac{1}{(1-\frac{ax}{2})^3}$$

$$\frac{F^{(k)}}{k!}=(\frac{a}{2})^{k+1}\frac{3.4...(k+2)}{k!}=(\frac{a}{2})^{k+1}\frac{(k+1)(k+2)}{2}$$

Similarly to the above derivation of the recurrent equations (the only difference is that here $v(0)=1$ and there are no A-terms), we get

$$g_j=\sum_{k=1}^{[\frac{j-1}{2}]}\sum_{p=0}^{k-1}\sum_{j_1+...+j_m=j-m-1}\frac{1}{j}\frac{1}{k!}F^{(k)}(x_i(0))C_k^p v^p(0)\frac{g_{j_1}}{j_1+1}...\frac{g_{j_{k-p}}}{j_{k-p}+1}$$

for g_j defined above. Here, as above, $\sum_{j_1+...+j_m=j-m-1}$ is the summation over all $j_1,...,j_{k-p}$ such that

$$j_1+...+j_{k-p}=j-k-1$$

As all coefficients are positive, then, neglecting the terms with $p>0$, we have for any $a>0$

$$\frac{1}{j}\sum_{k=1}^{[\frac{j-1}{2}]}\sum_{j_1+...+j_m=j-m-1}(\frac{a}{2})^{k+1}\frac{g_{j_1}}{j_1+1}...\frac{g_{j_k}}{j_k+1}\leq$$

$$\leq\frac{1}{j}\sum_{k=1}^{[\frac{j-1}{2}]}\sum_{j_1+...+j_m=j-m-1}(\frac{a}{2})^{k+1}\frac{(k+1)(k+2)}{2}\frac{g_{j_1}}{j_1+1}...\frac{g_{j_k}}{j_k+1}\leq g_j$$

Majorization From the recurrent formula for c_{ij} one can see that they can be written as

$$c_{ij} = \sum_{r=1}^{d_{ij}} b_{i,j,r} N^{I_{i,j,r}} M_{i,j,r}$$

where $b_{i,j,r}$ amd d_{ij} are some numbers not depending neither on N nor on F, and $M_{i,j,r} \in \mathbf{A}$. By lemma 2

$$\deg M_{i,j,r} \le j-1$$

We need other preliminary definitions. For any polynomial

$$P = \sum b_r N^{I_r} M_r$$

where b_r are the numbers, not depending neither on N nor on F, $M_r \in \mathbf{A}$, we put

$$\beta(P) = \sum_r |b_r| N^{I_r + Q(M_r)} C_F^{Q_0(M_r) - Q(M_r) + \deg M_r}$$

where, for any monomial M_r, the natural number $Q_0(M_r)$ equals the sum $\sum k$ over all factors (variables) $F_{i,k,q}$. In particular,

$$\beta(c_{ij}) = \sum_r |b_{i,j,r}| N^{I_{i,j,r} + Q(M_{i,j,r})} C_F^{Q_0(M_{i,j,r}) - Q(M_{i,j,r}) + \deg M_{i,j,r}}$$

By definition for any i, k

$$\beta(F_{i,k,0}) = C_F^{k+1}, \beta(\nabla^{\pm} F_{i,k,0}) = \beta(F_{i,k,1}) = C_F^{k+2} N^{-1} = C_F N^{-1} \beta(F_{i,k})$$

Moreover, for any two polynomials P_1, P_2

$$\beta(P_1 + P_2) \le \beta(P_1) + \beta(P_2), \beta(P_1 P_2) \le \beta(P_1)\beta(P_2) \tag{24}$$

and for any monomial M

$$\beta(\nabla^{\pm} M) \le (\deg M) N^{Q(M)-1} C_F^{Q_0(M) - Q(M) + \deg M + 1} = (\deg M) C_F N^{-1} \beta(M) \tag{25}$$

It follows that for any polynomial P

$$\beta(\nabla^{\pm} P) \le (\deg P) C_F N^{-1} \beta(P) \tag{26}$$

We call $\beta(P)$ the majorant of P as by

$$|F_{i,k,1}| = |\nabla^+ F_{i,k}| \le \int_{x_i}^{x_{i+1}} |F_{i,k+1}(x)| dx \le C_F^{k+2} N^{-1} = \beta(F_{i,k,1})$$

we have

$$|P| \le \beta(P)$$

From (24) and (17) it follows that

$$\beta(c_{ij}) \le \sum_{m=1}^{[\frac{j-1}{2}]} \sum_{j_1 + \ldots + j_m = j-m-1} \beta(A_{ij}(m; j_1, \ldots, j_m)) + \sum_{k=1}^{[\frac{j-1}{2}]} \sum_{j_1 + \ldots + j_k = j-k-1} \beta(B_{ij}(k; j_1, \ldots, j_k))$$

Our inductive hypothesis will be (c $g_j = g_j(1)$)

$$\beta(c_{ij}) \le \chi^j N^{\frac{5}{6}j - \frac{3}{2}} g_j, j = 1, 2, \ldots, J-2 \tag{27}$$

Initial data One can choose $\chi_0 > 0$ so that for $j - 1, 2, 3, 4$

$$\chi_0^j N^{\frac{5}{6}J-\frac{3}{2}} g_j \geq \beta(c_{ij})$$

In fact, only for $j = 3$ there is dependence on N, but $\frac{5}{6}3 - \frac{3}{2}$ is exactly N.

Inductive step for A-terms with $m > 1$ To estimate A-terms we distinguish two cases: $m = 1$ and $m > 1$. For $m > 1$ we use obvious bounds

$$\beta(\nabla^{\pm} c_{ij}) \leq 2\beta(c_{ij}), \beta(\nabla^{-}(\nabla^{+} c_{i,j_1} ... \nabla^{+} c_{i,j_m})) \leq 2^{m+1} \prod_p \beta(c_{i,j_p})$$

Then from (24) and (20) we get

$$\beta(A_{iJ}(m; j_1, ..., j_m)) \leq \frac{m+1}{J} N^{2+m} 2^{m+1} \frac{\beta(c_{i,j_1})}{j_1 + 1} ... \frac{\beta(c_{i,j_m})}{j_m + 1} \leq$$

$$\leq \frac{m+1}{J} N^{2+m} 2^{m+1} \chi^{J-m-1} N^{\frac{5}{6}(J-m-1)-m\frac{3}{2}} \prod_{p=1}^{m} \frac{g_{j_p}}{j_p + 1} \leq$$

$$\leq 2^{m+1} \chi^{J-m-1} N^{\frac{5}{6}J-\frac{3}{2}} \frac{m+1}{J} (\prod_{p=1}^{m} \frac{g_{j_p}}{j_p + 1})$$

as the exponent over N for $m \geq 2$ has the bound

$$2 + m + \frac{5}{6}(J - m - 1) - m\frac{3}{2} = \frac{5}{6}J - m\frac{8}{6} + \frac{7}{6} \leq \frac{5}{6}J - \frac{3}{2}$$

Then by lemma 4 with $a = 2$ (if $\chi \geq 2$)

$$\sum_{m=2}^{[\frac{j-1}{2}]} \sum_{j_1+...+j_m=j-m-1} \beta(A_{iJ}(m; j_1, ..., j_m)) \leq$$

$$\leq N^{\frac{5}{6}J-\frac{3}{2}} \chi^J \sum_{m=2}^{\frac{j-1}{2}} 2^{m+1} \chi^{-m-1} \sum_{j_1+...+j_m=j-m-1} \frac{m+1}{J} (\prod_{p=1}^{m} \frac{g_{j_p}}{j_p + 1}) \leq$$

$$\leq (\frac{2}{\chi})^3 N^{\frac{5}{6}J-\frac{3}{2}} \chi^J g_j \tag{28}$$

Inductive step for A-terms with $m = 1$ In case $m = 1$ we shall use the following bounds for $j = J - 2 \geq 3$. By (26), (27) and lemma 2

$$\beta(\nabla^{+} c_{ij}) \leq (j - 1)) N^{-1} C_F \beta(c_{ij}) \leq (j - 1)) N^{-1} C_F \chi^j N^{\frac{5}{6}j-\frac{3}{2}} g_j$$

Similarly

$$\beta(\nabla^{-}\nabla^{+} c_{i,j})| \leq ((j - 1)C_F)^2 N^{-2} \chi^j N^{\frac{5}{6}j-\frac{3}{2}} g_j$$

This gives additional summand (-2) in the exponent over N, equal to

$$3 - 2 + \frac{5}{6}(J - 2) - \frac{3}{2} \leq \frac{5}{6}J - \frac{3}{2}$$

that is

$$A_{ij}(1; j_1) = A_{ij}(1; J - 2) = \frac{1}{j} |d_1| N^3 \frac{\nabla^{-}\nabla^{+} c_{i,J-2}}{J - 1} \leq$$

$$\leq 2C_F^2 \chi^{j-2} N^{\frac{5}{6}J-\frac{3}{2}} g_{j-2} \leq \frac{2C_F^2 g_{j-2}}{\chi^2 g_j} \chi^j N^{\frac{5}{6}J-\frac{3}{2}} g_j \tag{29}$$

Inductive step for B-terms For B-terms the inductive bound is easier, but here the degree of monomials increases

$$\beta(B_{ij}(k;j_1,...,j_k)) = \frac{1}{j}\frac{1}{k!}\beta(F^{(k)}(x_i(0))\frac{c_{i,j_1}}{j_1+1}...\frac{c_{i,j_k}}{j_k+1}) \leq$$

$$\leq \frac{1}{j}\frac{C_F^{k+1}}{k!}\frac{1}{j_1+1}...\frac{1}{j_k+1}\beta(c_{i,j_1})...\beta(c_{i,j_k}) \leq \frac{1}{j}\frac{C_F^{k+1}}{k!}\frac{g_{j_1}}{j_1+1}...\frac{g_{j_k}}{j_k+1}\chi^{j_1+---+j_k}N^{j_1+---+j_k}$$

By lemma 4

$$\sum_{k=1}^{[\frac{j-1}{2}]}\sum_{j_1+...+j_k=j-k-1}\beta(B_{ij}(k;j_1,...,j_k)) \leq \frac{1}{j}\sum_{k=1}^{[\frac{j-1}{2}]}\frac{C_F^{k+1}}{k!}\sum_{j_1+...+j_k=j-k-1}\frac{g_{j_1}}{j_1+1}...\frac{g_{j_k}}{j_k+1}\chi^{J-k-1}N^{\frac{5}{7}j} \leq$$

$$\leq C_F e^{C_F}\chi^{J-2}N^{\frac{J}{2}}\frac{1}{j}\sum_{k=1}^{[\frac{j-1}{2}]}\sum_{j_1+...+j_k=j-k-1}\frac{g_{j_1}}{j_1+1}...\frac{g_{j_k}}{j_k+1} \leq C_F e^{C_F}\chi^{J-2}N^{\frac{J}{2}}g_j \qquad (30)$$

Finally, to end the proof of the theorem, we sum up three obtained expression (28),(29),(30), and choose $\chi = \chi_1 > 0$ so that

$$(8\chi^{-1} + \frac{2C_F^2 g_{j-2}}{g_j} + C_F e^{C_F})\chi^{-2} \leq 1$$

Then for any $\chi \geq \max(\chi_0, \chi_1)$ we will have

$$|c_{ij}| \leq \chi^J N^{\frac{J}{2}} g_j \leq \chi^J N^{\frac{J}{2}}$$

The theorem is proved.

References

[1] Goloubev V. V. Lectures on analytic theory of differential equations. Moscow, 1950 (in Russian).

[2] Lanford O. Time evolution of large classical systems. Lecture Notes in Physics, **38** (1975), 1-111. Springer.

[3] C. Cercignani, R. Illner, M. Pulvirenti. The mathematical theory of dilute gases. Springer. 1994.

[4] S. Caprino, M. Pulvirenti. A cluster expansion approach to a one-dimensional Boltzman equation: a validity result. Comm. Math. Phys., 1995, 166, 603-631.

[5] R. Dobrushin, J. Fritz. Nonequilibrium dynamis of one-dimensional infinite particle systems with a hard core interaction. Comm. Math. Phys., 1977, v. 55, 275-292.

[6] E. Caglioti, C. Marchioro, M. Pulvirenti. Non-equilibrium Dynamics of Three-Dimensional Infinite Particle System. Comm. Math. Phys., 2000, v. 215, No. 1, 25-43.

[7] Ya. Sinai. Construction of dynamics for one-dimensional systems of statistical mechanics. Theor. and Mathem. Physics, 11, 1972, No. 2, 248-258.

[8] H. Spohn. Large scale dynamics of interacting particles. 1991. Springer.

[9] V. A. Malyshev. Dynamical Clusters of Infinite Particle Dynamics. J. Math. Physics, 2005, v. 46, No. 7.

[10] Malyshev V. A. Why the current flows: multi-particle one-dimensional model. Theoretical and mathematical physics, 2008, v. 155, No. 2, pp. 301-311.

[11] V. A. Malyshev. Fixed points for one-dimensional particle systems with strong interaction. Moscow Math. Journal, 2012, v. 12, No.1.

[12] Malyshev V. A. Critical states of multi-particle systems with strong interaction on the circle. Problems of information transmission, 47:2 (2011), 117-127.